Functions

Fact — A function is a *mapping* from one set to another set.

$$\underbrace{f}_{function} : \underbrace{\mathbb{R}}_{domain} \rightarrow \underbrace{\mathbb{R}}_{codomain}$$
$$x \mapsto x^2$$

*The set of values which are mapped to is called the **image** or **range**. It is not necessarily the whole codomain*

$$range = image \subset codomain$$

DRAW PICUTRES OF DOMAINS and CODOMAINS ,define injective, surjective and bijective

Example

Find the range of the function $f(x) = x^2 - 2x + 4$ with domain $-1 \leq x \leq 2$.

Since $f(x) = (x-1)^2 + 3$, $f(x) \geq 3$ and $f(1) = 3$. Since $1 \in$ domain 3 is attained. A quick sketch shows us that $f(-1) = 7$, $f(2) = 4$ so the range is $[3, 7]$

Example

A function is defined as $f(x) = \frac{2x}{x^2+1}$.

(a) By letting $y = f(x)$, show that $yx^2 - 2x + y = 0$.

(b) Using the discriminant of a quadratic, find the range of values for y such that (a) has real roots.

(c) Deduce the range of the function $f(x)$ and hence sketch the curve $f(x)$

Composition of Functions

Suppose we have

$$f : A \to B$$
$$g : B \to C$$

then it makes sense to talk about $gf = g \circ f$, which is defined by:

$$gf : A \to C$$
$$a \mapsto g(\underbrace{f(a)}_{\in B})$$

If $A = B = C$, it can make sense to talk about both fg and gf, for example $\sin : \mathbb{R} \to \mathbb{R}, x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$, so $\sin(x^2)$ and $\sin^2(x)$ are both valid compositions of these functions.

However, if we have

$$f : \mathbb{R}_{<0} \to \mathbb{R}_{>0}$$
$$x \mapsto -x$$
$$g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$$
$$x \mapsto \sqrt{x}$$

we can have $gf(-4) = g(-(-4)) = g(4) = 2$, *but* $fg(-4)$ *is matherror.*

Example

The functions f and g are given by:

$$f(x) = 5e^{-x} + 1, x \in \mathbb{R}, x \geq 0$$
$$g(x) = 2x + 1, x \in \mathbb{R}$$

(a) Find...

(i) ... an expression for $gf(x)$

(ii) ... the range of $gf(x)$

(iii) ... the domain of $fg(x)$

(b) Show the only solution of the equation $fg(x) = 5e^{2x+1} - 9$ can be written as $x = \frac{1}{2}\left[-1 + \ln(1 + \sqrt{2})\right]$

(a) *(i)*

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(5e^{-x} + 1) \\ &= 2(5e^{-x} + 1) + 1 \\ &= 10e^{-x} + 3 \end{aligned}$$

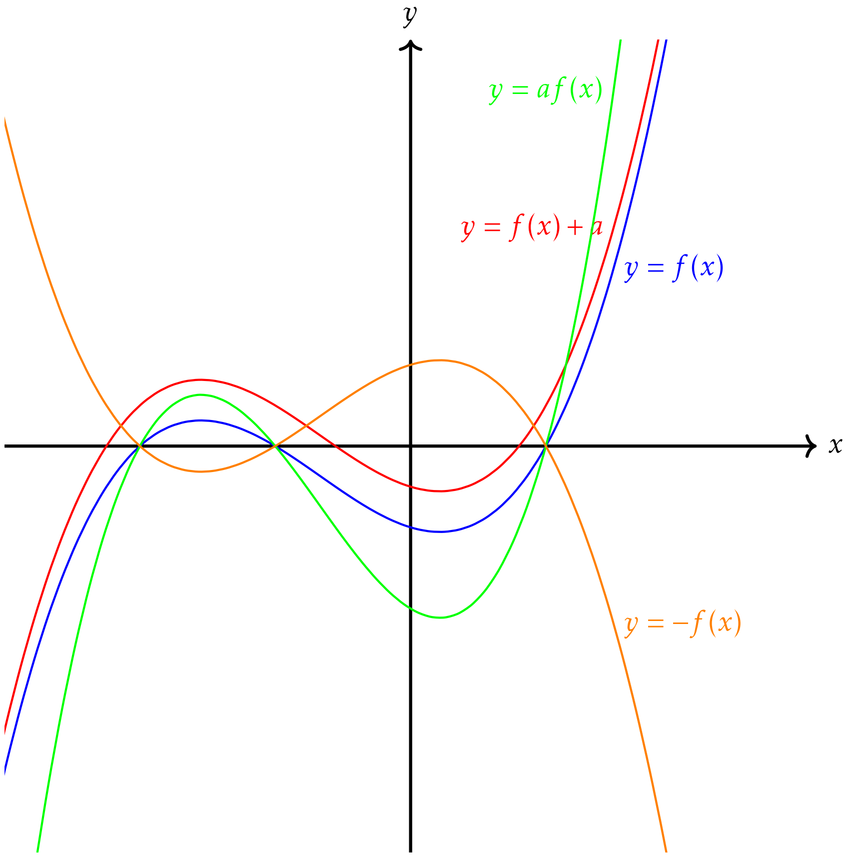
(ii) *The domain if f is $[0, \infty)$ so the domain of gf is $[0, \infty)$. Since gf is decreasing, the largest value is $gf(0) = 10e^{-0} + 3 = 13$. Since $e^{-x} \to 0$, we have $10e^{-x} \to 0$ and $10e^{-x} + 3 \to 3$, so the smallest value is 3. But we cannot achieve 3. Therefore our range is $(3, 13]$ or $3 < gf(x) \leq 13$.*

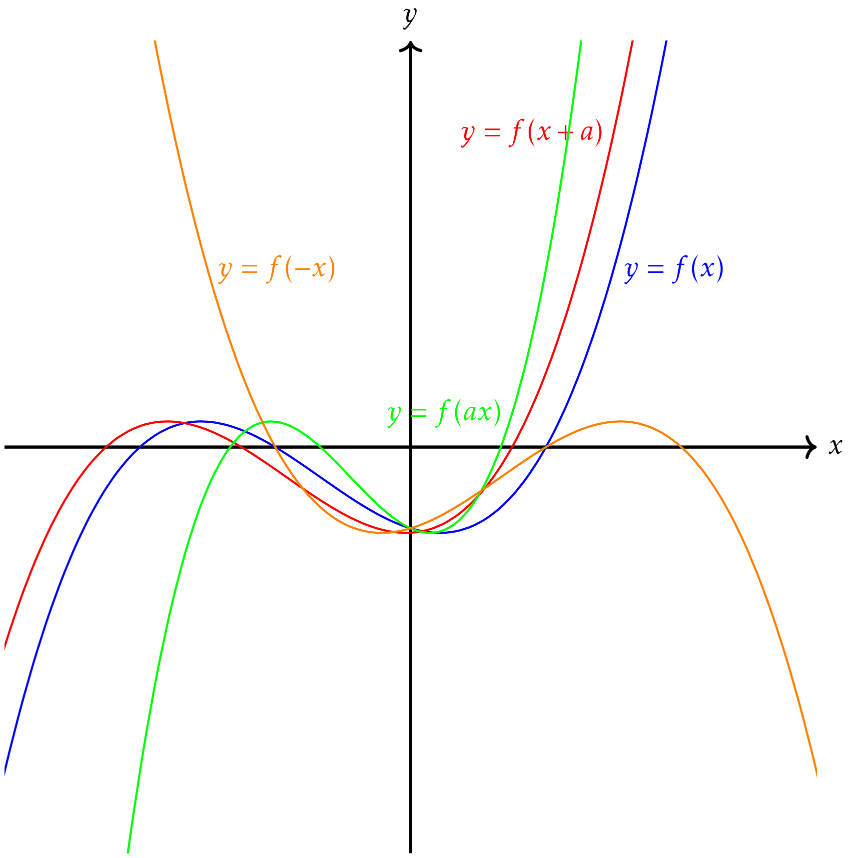
(iii) *$g(x)$ can take anything is input, but when it is passed into f it needs to be positive. Therefore $g(x) \geq 0$, ie $2x + 1 \geq 0$ or $x \geq -\frac{1}{2}$. Therefore the domain is $[-\frac{1}{2}, \infty)$*

(b) *$fg(x) = 5e^{-(2x+1)} + 1$ therefore,*

$$\begin{aligned} && 5e^{-(2x+1)} + 1 &= 5e^{2x+1} - 9 \\ \Rightarrow && 5 + y &= 5y^2 - 9y \\ \Rightarrow && 0 &= 5y^2 - 10y - 5 \\ && &= 5(y^2 - 2y - 1) \\ \Rightarrow && y &= 1 \pm \sqrt{2} \\ \underbrace{\Rightarrow}_{y>0} && y &= 1 + \sqrt{2} \\ \Rightarrow && e^{2x+1} &= 1 + \sqrt{2} \\ \Rightarrow && 2x + 1 &= \ln(1 + \sqrt{2}) \\ \Rightarrow && x &= \frac{1}{2}\left(-1 + \ln(1 + \sqrt{2})\right) \end{aligned}$$

Transformation of functions





$f(x) \mapsto$	Transformation
$f(x+a)$	translation by $\binom{-a}{0}$
$f(x)+a$	translation by $\binom{0}{a}$
$f(ax)$	stretch by scale factor $\frac{1}{a}$ parallel to x-axis
$af(x)$	stretch by scale factor a parallel to y-axis
$f(-x)$	reflection in y-axis
$-f(x)$	reflection in x-axis